Some Results on Homomorphism of Fuzzy Hv-Submodules

Manoj Kumar Dewangan and Pranjali Sharma
Department of Mathematics
Shri Shankaracharya Institute of Professional Management and Technology, Raipur, Chhattisgarh

Abstract
Atanassov introduced the notion of intuitionistic fuzzy sets as a generalization of the notion of fuzzy sets. Using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" of fuzzy points with fuzzy sets, the concept of a homomorphism of an interval-valued intuitionistic (α, β)-fuzzy H_v-submodule is considered and some interesting properties are investigated, where $\alpha \in \{\in, q\}$, $\beta \in \{\in, q, \in \vee q, \in \wedge q\}$.

Keywords: Hyperstructure, H_v-module, Fuzzy set, Intuitionistic fuzzy set, Interval valued intuitionistic (α, β)-fuzzy H_v-submodule.

Mathematics Subject Classification: 20N20

1. Introduction

The concept of hyperstructure was introduced in 1934 by Marty [4]. Hyperstructures have many applications to several branches of pure and applied sciences. Vougiouklis [13] introduced the notion of H_v-structures, and Davaz [1] surveyed the theory of H_v-structures. After the introduction of fuzzy sets by Zadeh [8], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [5] is one among them. For more details on intuitionistic fuzzy sets, we refer the reader to [6, 7].

The idea of quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [10], played a vital role to generate some different types of fuzzy subgroups. Bhakat and Das [11, 12] gave the concepts of (α, β)-fuzzy subgroups by using the notion of "belongingness $(\in)$" and "quasi-coincidence (q)" between a fuzzy point and a fuzzy subgroup, where α, β are any two of $\{\in, q, \in \vee q, \in \wedge q\}$ with $\alpha \neq \in \wedge q$, and introduced the concept of an $(\in, \in \vee q)$-fuzzy subgroup. In [15] Yuan, Li et al. redefined (α, β)-intuitionistic fuzzy subgroups. M. Asghari-Larimi [9] gave the concept of a homomorphism of intuitionistic (α, β)-fuzzy H_v-submodules. Basing on [9], in this paper, we introduce the concept of a homomorphism of an interval valued intuitionistic (α, β)-fuzzy H_v-submodule of an H_v-module and describe the characteristic properties.

The paper is organized as follows: in section 2 some fundamental definitions on H_v-structures and fuzzy sets are explored, in section 3 we give some results on an interval valued intuitionistic (α, β)-fuzzy H_v-submodules and in section 4 we introduce the concept of a homomorphism of an interval valued intuitionistic (α, β)-fuzzy H_v-submodule of an H_v-module and establish some useful results.

2. Basic Definitions

We first give some basic definitions for proving the further results.

Definition 2.1 [3] Let X be a non-empty set. A mapping $\mu : X \to [0,1]$ is called a fuzzy set in X. The complement of μ, denoted by μ^c, is the fuzzy set in X given by $\mu^c(x) = 1 - \mu(x) \quad \forall x \in X$.

Definition 2.2 [3] An intuitionistic fuzzy set A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$, where the functions $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership and degree of non membership of each element $x \in X$ to the set A respectively and

$0 \le \mu_A(x) + \lambda_A(x) \le 1$ for all $x \in X$. We shall use the symbol $A = \{\mu_A, \lambda_A\}$ for the intuitionistic fuzzy set $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$.

Definition 2.3 [3] Let $A = \{\mu_A, \lambda_A\}$ and $B = \{\mu_B, \lambda_B\}$ be intuitionistic fuzzy sets in X. Then

(1) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x) \text{ and } \lambda_A(x) \le \lambda_B(x)$,

(2) $A^c = \{(x, \lambda_A(x), \mu_A(x)) : x \in X\}$,

(3) $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \max\{\lambda_A(x), \lambda_B(x)\}) : x \in X\}$,

(4) $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \min\{\lambda_A(x), \lambda_B(x)\}) : x \in X\}$,

(5) $\Box A = \{(x, \mu_A(x), \mu_A^c(x)) : x \in X\}$,

(6) $\Diamond A = \{(x, \lambda_A^c(x), \lambda_A(x)) : x \in X\}$.

Definition 2.4 [14] Let G be a non–empty set and $* : G \times G \to \wp^*(G)$ be a hyperoperation, where $\wp^*(G)$ is the set of all the non-empty subsets of G. Where $A * B = \bigcup_{a \in A, b \in B} a * b, \ \forall A, B \subseteq G$. The $*$ is called weak commutative if $x * y \cap y * x \ne \phi, \quad \forall x, y \in G$. The $*$ is called weak associative if $(x * y) * z \cap x * (y * z) \ne \phi, \quad \forall x, y, z \in G$.

A hyperstructure $(G, *)$ is called an H_v–group if

(i) $*$ is weak associative.

(ii) $a * G = G * a = G, \quad \forall a \in G$ (Reproduction axiom).

Definition 2.5 [14] An H_v–ring is a system $(R, +, \cdot)$ with two hyperoperations satisfying the ring-like axioms:

(i) $(R, +, \cdot)$ is an H_v-group, that is,

$((x + y) + z) \cap (x + (y + z)) \ne \phi \quad \forall x, y \in R,$

$a + R = R + a = R \quad \forall a \in R;$

(ii) $(R, \cdot)$ is an H_v-semigroup;

(iii) $(\cdot)$ is weak distributive with respect to $(+)$, that is, for all $x, y, z \in R$,

$(x \cdot (y + z)) \cap (x \cdot y + x \cdot z) \ne \phi,$

$((x + y) \cdot z) \cap (x \cdot z + y \cdot z) \ne \phi.$

Definition 2.6 [2] Let R be an H_v-ring. A nonempty subset I of R is called a left (resp., right) H_v-ideal if the following axioms hold:

(i) $(I, +)$ is an H_v-subgroup of $(R, +)$,

(ii) $R \cdot I \subseteq I$ (resp., $I \cdot R \subseteq I$).

Definition 2.7 [2] Let $(R, +, \cdot)$ be an H_v–ring and μ a fuzzy subset of R. Then μ is said to be a left (resp., right) fuzzy H_v-ideal of R if the following axioms hold:

(1) $\min\{\mu(x), \mu(y)\} \le \inf\{\mu(z) : z \in x + y\} \forall x, y \in R,$

(2) For all $x, a \in R$ there exists $y \in R$ such that $x \in a + y$ and $\min\{\mu(a), \mu(x)\} \le \mu(y),$

(3) For all $x, a \in R$ there exists $z \in R$ such that $x \in z + a$ and $\min\{\mu(a), \mu(x)\} \le \mu(z),$

(4) $\mu(y) \le \inf\{\mu(z) : z \in x \cdot y\}$ respectively $\mu(x) \le \inf\{\mu(z) : z \in x \cdot y\} \quad \forall x, y \in R.$

Definition 2.8 [2] An intuitionistic fuzzy set $A=\{\mu_A,\lambda_A\}$ in R is called a left (resp., right) intuitionistic fuzzy H_v-ideal of R if following axioms hold:

(1) $\min\{\mu_A(x),\mu_A(y)\}\le\inf\{\mu_A(z):z\in x+y\}\forall x,y\in R,$

(2) For all $x,a\in R$ there exists $y,z\in R$ such that $x\in(a+y)\cap(z+a)$ and $\min\{\mu_A(a),\mu_A(x)\}\le\min\{\mu_A(y),\mu_A(z)\},$

(3) $\mu_A(y)\le\inf\{\mu_A(z):z\in x\cdot y\}$ respectively $\mu_A(x)\le\inf\{\mu_A(z):z\in x\cdot y\}\quad\forall x,y\in R,$

(4) $\sup\{\lambda_A(z):z\in x+y\}\le\max\{\lambda_A(x),\mu_A(y)\}\forall x,y\in R,$

(5) For all $x,a\in R$ there exists $y,z\in R$ such that $x\in(a+y)\cap(z+a)$ and $\max\{\lambda_A(y),\lambda_A(z)\}\le\max\{\lambda_A(a),\lambda_A(x)\},$

(6) $\sup\{\lambda_A(z):z\in x\cdot y\}\le\lambda_A(y)$ respectively $\sup\{\lambda_A(z):z\in x\cdot y\}\le\lambda_A(x)\quad\forall x,y\in R.$

Definition 2.9 [17] A nonempty set M is called an H_v-module over an H_v -ring R if (M, +) is a weak commutative H_v -group and there exists a map

$.:R\times M\to\wp^*(M),(r,x)\to r.x$ Such that for all $a,b\in R$ and $x,y\in M$, we have

$(a.(x+y))\cap(a.x+a.y)\ne\phi,$

$((x+y).a)\cap(x.a+y.a)\ne\phi,$

$(a.(b.x))\cap((a.b).x)\ne\phi.$

Note that by using fuzzy sets, we can consider the structure of H_v -module on any ordinary module which is a generalization of a module.

Definition 2.10 [12] A fuzzy set μ in M is called a fuzzyH_v-submodule of M if

(1) $\min\{\mu(x),\mu(y)\}\le\inf\{\mu(z):z\in x+y\}\forall x,y\in M,$

(2) For all $x,a\in M$ there exists $y,z\in M$ such that $x\in(a+y)\cap(z+a)$ and $\min\{\mu(a),\mu(x)\}\le\inf\{\mu(y),\mu(z)\},$

(3) $\mu(y)\le\inf\{\mu(z):z\in x\cdot y\}$ for all $y\in M$ and $x\in R.$

Definition 2.11 [16] An intuitionistic fuzzy set $A=\{\mu_A,\lambda_A\}$ in an H_v –module M over an H_v –ring R is said to be an intuitionistic fuzzy H_v-submodule of M if the following axioms hold:

(1) $\min\{\mu_A(x),\mu_A(y)\}\le\inf\{\mu_A(z):z\in x+y\}$ and $\max\{\lambda_A(x),\lambda_A(y)\}\ge\sup\{\lambda_A(z):z\in x+y\}$ for all $x,y\in M,$

(2) For all $x,a\in M$ there exists $y\in M$ such that $x\in a+y$ and $\min\{\mu_A(a),\mu_A(x)\}\le\mu_A(y)$ and $\max\{\lambda_A(a),\lambda_A(x)\}\ge\lambda_A(y),$

(3) For all $x,a\in M$ there exists $z\in M$ such that $x\in z+a$ and $\min\{\mu_A(a),\mu_A(x)\}\le\mu_A(z)$ and $\max\{\lambda_A(a),\lambda_A(x)\}\ge\lambda_A(z),$

(4) $\mu_A(x)\le\inf\{\mu_A(z):z\in r\cdot x\}$ and $\lambda_A(x)\ge\sup\{\lambda_A(z):z\in r\cdot x\}$ for all $x\in M$ and $r\in R.$

Definition 2.12 [12] Let μ be a fuzzy subset of R. If there exist a $t\in(0,1]$ and an $x\in R$ such that

$$\mu(y)=\begin{cases} t & if\ y=x \\ 0 & if\ y\neq x \end{cases}$$

Then μ is called a fuzzy point with support x and value t and is denoted by x_t.

Definition 2.13 [12] Let μ be a fuzzy subset of R and x_t be a fuzzy point. (1) If $\mu(x)\geq t$, then we say x_t belongs to μ, and write $x_t\in\mu$.

(2) If $\mu(x)+t>1$, then we say x_t is quasi-coincident with μ, and write $x_t q\mu$.

$(3) x_t\in\vee q\mu \Leftrightarrow x_t\in\mu\ or\ x_t q\mu.$.

$(4) x_t\in\wedge q\mu \Leftrightarrow x_t\in\mu\ and\ x_t q\mu.$

In what follows, unless otherwise specified, α and β will denote any one of $\in, q, \in\vee q$ or $\in\wedge q$ with $\alpha\neq\in\wedge q$, which was introduced by Bhakat and Das [9].

By an interval number $\tilde{a}$ we mean an interval $\left[a^-,a^+\right]$ where $0\leq a^-\leq a^+\leq 1$. The set of all interval numbers is denoted by $D[0,1]$. We also identify the interval $[a,a]$ by the number $a\in[0,1]$.

For the interval numbers $\tilde{a}_i=\left[a_i^-,a_i^+\right]\in D[0,1], i\in I$, we define

$$\max\left\{\tilde{a}_i,\tilde{b}_i\right\}=\left[\max\left(a_i^-,b_i^-\right),\max\left(a_i^+,b_i^+\right)\right],$$

$$\min\left\{\tilde{a}_i,\tilde{b}_i\right\}=\left[\min\left(a_i^-,b_i^-\right),\min\left(a_i^+,b_i^+\right)\right],$$

$$\inf\tilde{a}_i=\left[\bigwedge_{i\in I}a_i^-,\bigwedge_{i\in I}a_i^+\right],\sup\tilde{a}_i=\left[\bigvee_{i\in I}a_i^-,\bigvee_{i\in I}a_i^+\right]$$

and put

$(1)\tilde{a}_1\leq\tilde{a}_2\Leftrightarrow a_1^-\leq a_2^-$ and $a_1^+\leq a_2^+$,

$(2)\tilde{a}_1=\tilde{a}_2\Leftrightarrow a_1^-=a_2^-$ and $a_1^+=a_2^+$,

$(3)\tilde{a}_1<\tilde{a}_2\Leftrightarrow\tilde{a}_1\leq\tilde{a}_2$ and $\tilde{a}_1\neq\tilde{a}_2$,

$(4)k\tilde{a}=\left[ka^-,ka^+\right]$, whenever $0\leq k\leq 1$.

It is clear that $(D[0,1],\leq,\vee,\wedge)$ is a complete lattice with $0=[0,0]$ as least element and $1=[1,1]$ as greatest element.

By an interval valued fuzzy set F on X we mean the set $F=\left\{\left(x,\left[\mu_F^-(x),\mu_F^+(x)\right]\right):x\in X\right\}$. Where μ_F^- and μ_F^+ are fuzzy subsets of X such that $\mu_F^-(x)\leq\mu_F^+(x)$ for all $x\in X$. Put $\tilde{\mu}_F(x)=\left[\mu_F^-(x),\mu_F^+(x)\right]$. Then $F=\left\{\left(x,\tilde{\mu}_F(x)\right):x\in X\right\}$, where $\tilde{\mu}_F:X\to D[0,1]$.

If A, B are two interval valued fuzzy subsets of X, then we define

$A\subseteq B$ if and only if for all $x\in X$, $\mu_A^-(x)\leq\mu_B^-(x)$ and $\mu_A^+(x)\leq\mu_B^+(x)$,

$A=B$ if and only if for all $x\in X$, $\mu_A^-(x)=\mu_B^-(x)$ and $\mu_A^+(x)=\mu_B^+(x)$.

Also, the union, intersection and complement are defined as follows: let A; B be two interval valued fuzzy subsets of X, then

$$A \cup B = \left\{ \left(x, \left[\max\left\{ \mu_A^-(x), \mu_B^-(x) \right\}, \max\left\{ \mu_A^+(x), \mu_B^+(x) \right\} \right] \right) : x \in X \right\},$$

$$A \cap B = \left\{ \left(x, \left[\min\left\{ \mu_A^-(x), \mu_B^-(x) \right\}, \min\left\{ \mu_A^+(x), \mu_B^+(x) \right\} \right] \right) : x \in X \right\},$$

$$A^c = \left\{ \left(x, \left[\left\{ 1 - \mu_A^-(x), 1 - \mu_A^+(x) \right\} \right] \right) : x \in X \right\}.$$

According to Atanassov an interval valued intuitionistic fuzzy set on X is defined as an object of the form $A = \left\{ \left(x, \tilde{\mu}_A(x), \tilde{\lambda}_A(x) \right) : x \in X \right\}$, where $\tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(x)$ are interval valued fuzzy sets on X such that $0 \le \sup \tilde{\mu}_A(x) + \sup \tilde{\lambda}_A(x) \le 1$ for all $x \in X$.

For the sake of simplicity, in the following such interval valued intuitionistic fuzzy sets will be denoted by $A = \left(\tilde{\mu}_A, \tilde{\lambda}_A \right)$.

3. Interval Valued Intuitionistic (α, β) -fuzzy Hv-submodules

In this section we give some results on interval valued intuitionistic (α, β)-fuzzy Hv-submodule.

Definition 3.1. [18] An interval valued intuitionistic fuzzy set $A = \{\tilde{\mu}_A, \tilde{\lambda}_A\}$ in M is called an interval valued intuitionistic (α, β)-fuzzy H_v-submodule of M if for all $t, r \in (0,1]$,

$(1)\ \forall x, y \in M, \quad x_t \alpha \tilde{\mu}_A, y_r \alpha \tilde{\mu}_A \Rightarrow z_{t \wedge r} \beta \tilde{\mu}_A$ for all $z \in x + y$,

$(2)\ \forall x, a \in M, \quad x_t \alpha \tilde{\mu}_A, a_r \alpha \tilde{\mu}_A \Rightarrow (y \wedge z)_{t \wedge r} \beta \tilde{\mu}_A$ for some $y, z \in M$ with $x \in (a + y) \cap (z + a)$,

$(3)\ \forall x, y \in M, \quad y_t \alpha \tilde{\mu}_A \Rightarrow z_t \beta \tilde{\mu}_A$ for all $z \in x \cdot y$,

$(4)\ \forall x, y \in M, \quad x_t \bar{\alpha} \tilde{\lambda}_A, y_r \bar{\alpha} \tilde{\lambda}_A \Rightarrow z_{t \wedge r} \bar{\beta} \tilde{\lambda}_A$ for all $z \in x + y$,

$(5)\ \forall x, a \in M, \quad x_t \bar{\alpha} \tilde{\lambda}_A, a_r \bar{\alpha} \tilde{\lambda}_A \Rightarrow (y \wedge z)_{t \wedge r} \bar{\beta} \tilde{\lambda}_A$ for some $y, z \in M$ with $x \in (a + y) \cap (z + a)$,

$(6)\ \forall x, y \in M, \quad y_t \bar{\alpha} \tilde{\lambda}_A \Rightarrow z_{t \wedge r} \bar{\beta} \tilde{\lambda}_A$ for all $z \in x \cdot y$.

Lemma 3.2. [18]Let $A = \{\tilde{\mu}_A, \tilde{\lambda}_A\}$ be an interval valued intuitionistic fuzzy set in M. Then for all $x \in M$ and $t \in (0,1]$, we have

$(1)\ x_t q \tilde{\mu}_A \Leftrightarrow x_t \,\bar{\in}\, \tilde{\mu}_A^c.$

$(2)\ x_t \in \vee q \tilde{\mu}_A \Leftrightarrow x_t \overline{\in \wedge q}\ \tilde{\mu}_A^c.$

Proof. (1) Let $x \in M$ and $t \in (0,1]$. Then, we have

$$\begin{aligned} x_t q \tilde{\mu}_A &\Leftrightarrow \tilde{\mu}_A(x) + t > 1 \\ &\Leftrightarrow 1 - \tilde{\mu}_A(x) < t \\ &\Leftrightarrow \tilde{\mu}_A^c(x) < t \\ &\Leftrightarrow x_t \,\bar{\in}\, \tilde{\mu}_A^c. \end{aligned}$$

(2) Let $x \in M$ and $t \in (0,1]$. Then, we have

$x_t \in \vee q \tilde{\mu}_A \Leftrightarrow x_t \in \tilde{\mu}_A$ or $x_t q \tilde{\mu}_A$

$$\Leftrightarrow \tilde{\mu}_A(x) \geq t \text{ or } \tilde{\mu}_A(x) + t > 1$$
$$\Leftrightarrow 1 - \tilde{\mu}_A^c(x) \geq t \text{ or } 1 - \tilde{\mu}_A^c(x) + t > 1$$
$$\Leftrightarrow x_t \overline{q} \tilde{\mu}_A^c \text{ or } x_t \overline{\in} \tilde{\mu}_A^c$$
$$\Leftrightarrow x_t \overline{\in \wedge q} \tilde{\mu}_A^c.$$

Definition 3.3. Let $t \in [0,1]$ and μ is a fuzzy set in M. Then, the set $U(\alpha\mu, t) = \{x \in M : x_t \alpha \mu\}$ (respectively, $L(\alpha\mu, t) = \{x \in M : x_t \overline{\alpha} \mu\}$) is called an upper (respectively, lower) t -level cut of $\alpha\mu$.

Theorem 3.4. Let $A = \{\tilde{\mu}_A, \tilde{\lambda}_A\}$ is an interval-valued intuitionistic (α, β) -fuzzy H_v -submodule of M, then the set $U(\alpha\tilde{\mu}_A, t)(U(\alpha'\tilde{\mu}_A, t))$ is an H_v -submodule of Mfor all $t \in \operatorname{Im}(\tilde{\mu}_A)$ Where $(\alpha, \beta) \in \{(\in, \in), (q, q), (\in, \in \wedge q), (q, \in \wedge q)\}$ $((\alpha, \beta) \in \{(\in, \in \wedge q), (q, \in \wedge q)\})$.

Proof. We only prove the case of $(\alpha, \beta) = (\in, \in \wedge q)$. The others are analogous.

We must show that

$(i)\, a + U(\in \tilde{\mu}_A, t) = U(\in \tilde{\mu}_A, t) + a = U(\in \tilde{\mu}_A, t)$ for all $U(\in \tilde{\mu}_A, t)$,

$(ii)\, R.U(\in \tilde{\mu}_A, t) \subseteq U(\in \tilde{\mu}_A, t)$.

Case (i). Suppose that $t \in \operatorname{Im}(\mu_A) \subseteq [0,1]$ and let $a, x \in U(\in \tilde{\mu}_A, t)$. By definition, we have $a_t \in \tilde{\mu}_A$ and $x_t \in \tilde{\mu}_A$. Hence $\tilde{\mu}_A(a) \geq t$ and $\tilde{\mu}_A(x) \geq t$. Since $\tilde{\mu}_A$ is an $(\in, \in \wedge q)$-fuzzy H_v -submodule of M. It follows from condition (1) of Definition 3.1 that $z_t \in \wedge q \tilde{\mu}_A$ for all $z \in a + x$ and $z \in x + a$. Which implies $z_t \in \tilde{\mu}_A$ for all $z \in a + x$ and $z \in x + a$. Therefore $a + x \subseteq U(\in \tilde{\mu}_A, t)$ and $x + a \subseteq U(\in \tilde{\mu}_A, t)$.

On the other hand, since $a, x \in U(\in \tilde{\mu}_A, t)$. Thus By condition (2) of Definition 3.1, we have $(y \wedge z)_t \in \wedge q \tilde{\mu}_A$ for some $y, z \in M$ with $x \in (a + y) \cap (z + a)$, which implies $y_t \in \tilde{\mu}_A$ and $z_t \in \tilde{\mu}_A$. Thus $y \in U(\in \tilde{\mu}_A, t)$ and $z \in U(\in \tilde{\mu}_A, t)$. This proves that $U(\in \tilde{\mu}_A, t) \subseteq a + U(\in \tilde{\mu}_A, t)$ and $U(\in \tilde{\mu}_A, t) \subseteq U(\in \tilde{\mu}_A, t) + a$, for all $a \in U(\in \tilde{\mu}_A, t)$.

Case (ii). Let $x \in R, y \in U(\in \tilde{\mu}_A, t)$. Hence $y_t \in \tilde{\mu}_A$. Since $A = \{\tilde{\mu}_A, \tilde{\lambda}_A\}$ is an interval-valuedintuitionistic $(\in, \in \wedge q)$ -fuzzy H_v -submodule of M. It follows from conditions (3) of Definition 3.1 that $z_t \in \wedge q \tilde{\mu}_A$ for all $z \in x.y$. Thus $z_t \in \tilde{\mu}_A$. Then $z \in U(\in \tilde{\mu}_A, t)$ and so $x.y \subseteq U(\in \tilde{\mu}_A, t)$. Therefore $R.U(\in \tilde{\mu}_A, t) \subseteq U(\in \tilde{\mu}_A, t)$.

This completes the proof.

Corollary 3.5. Let $\tilde{\mu}_A$ is an interval-valued (α, β)-fuzzy H_v -submodule of M, then the set $U(\alpha\tilde{\mu}_A, t)(U(\alpha'\tilde{\mu}_A, t))$ is an H_v -submodule of Mfor all $t \in \operatorname{Im}(\mu_A)$ Where $(\alpha, \beta) \in \{(\in, \in), (q, q), (\in, \in \wedge q), (q, \in \wedge q)\}$ $((\alpha, \beta) \in \{(\in, \in \wedge q), (q, \in \wedge q)\})$.

Theorem 3.6. Let $A=\{\tilde{\mu}_A,\tilde{\lambda}_A\}$ is an interval-valued intuitionistic (α,β) -fuzzy H_v-submodule of M, then the set $L(\alpha\tilde{\lambda}_A,t)\left(L(\alpha'\tilde{\lambda}_A,t)\right)$ is an H_v-submodule of Mfor all $t\in\operatorname{Im}(\tilde{\lambda}_A)$, where $(\alpha,\beta)\in\{(\in,\in),(q,q),(\in,\in\wedge q),(q,\in\wedge q)\}$ $\left((\alpha,\beta)\in\{(\in,\in\wedge q),(q,\in\wedge q)\}\right)$.

Proof. We only prove the case of $(\alpha,\beta)=(\in,\in\wedge q)$. The others are analogous.

We must show that

$(i)\, a+L(\in\tilde{\lambda}_A,t)=L(\in\tilde{\lambda}_A,t)+a=L(\in\tilde{\lambda}_A,t)$ for all $L(\in\tilde{\lambda}_A,t)$,

$(ii)\, R.L(\in\tilde{\lambda}_A,t)\subseteq L(\in\tilde{\lambda}_A,t)$.

Case (i). Suppose that $t\in\operatorname{Im}(\tilde{\lambda}_A)\subseteq[0,1]$ and let $a,x\in L(\in\tilde{\lambda}_A,t)$. By definition, we have $a_t\,\overline{\in}\,\tilde{\lambda}_A$ and $x_t\,\overline{\in}\,\tilde{\lambda}_A$. Hence $\tilde{\lambda}_A(a)<t$ and $\tilde{\lambda}_A(x)<t$. Since $A=\{\tilde{\mu}_A,\tilde{\lambda}_A\}$ is an interval-valued intuitionistic $(\in,\in\vee q)$-fuzzy H_v -submodule of M. It follows from condition (4) of Definition 3.1 that $z_t\overline{\in\vee q}\tilde{\lambda}_A$ for all $z\in a+x$ and $z\in x+a$. Which implies $z_t\,\overline{\in}\,\tilde{\lambda}_A$ for all $z\in a+x$ and $z\in x+a$. Therefore $a+x\subseteq L(\in\tilde{\lambda}_A,t)$ and $x+a\subseteq L(\in\tilde{\lambda}_A,t)$. On the other hand, since $a,x\in L(\in\tilde{\lambda}_A,t)$. Thus $a_t,x_t\,\overline{\in}\,\tilde{\lambda}_A$. By condition (5) of Definition 3.1, we have $(y\wedge z)_t\overline{\in\vee q}\tilde{\lambda}_A$ for some $y,z\in M$ with $x\in(a+y)\cap(z+a)$, which implies $y_t\,\overline{\in}\,\tilde{\lambda}_A$ and $z_t\,\overline{\in}\,\tilde{\lambda}_A$. Thus $y\in L(\in\tilde{\lambda}_A,t)$ and $z\in L(\in\tilde{\lambda}_A,t)$. This proves that $L(\in\tilde{\lambda}_A,t)\subseteq a+L(\in\tilde{\lambda}_A,t)$ and $L(\in\tilde{\lambda}_A,t)\subseteq L(\in\tilde{\lambda}_A,t)+a$, for all $a\in L(\in\tilde{\lambda}_A,t)$.

Case (ii). Let $x\in R, y\in L(\in\tilde{\lambda}_A,t)$. Hence $y_t\,\overline{\in}\,\tilde{\lambda}_A$. Since $\tilde{\lambda}_A$ is an anti interval-valuedintuitionistic $(\in,\in\vee q)$ -fuzzy H_v-submodule of M. It follows from conditions (6) of Definition 3.1 that $z_t\overline{\in\vee q}\tilde{\lambda}_A$ for all $z\in x.y$.

Then $z\in L(\in\tilde{\lambda}_A,t)$ and so $x.y\subseteq L(\in\tilde{\lambda}_A,t)$. Therefore $R.L(\in\tilde{\lambda}_A,t)\subseteq L(\in\tilde{\lambda}_A,t)$.

This completes the proof.

Corollary 3.7. Let $\tilde{\lambda}_A$ is an anti interval-valued (α,β)-fuzzy H_v-submodule of M, then the set $L(\alpha\tilde{\lambda}_A,t)\left(L(\alpha'\tilde{\lambda}_A,t)\right)$ is an H_v-submodule of Mfor all $t\in\operatorname{Im}(\tilde{\lambda}_A)$, Where $(\alpha,\beta)\in\{(\in,\in),(q,q),(\in,\in\wedge q),(q,\in\wedge q)\}$ $\left((\alpha,\beta)\in\{(\in,\in\wedge q),(q,\in\wedge q)\}\right)$.

4. Homomorphism of Interval-valued Intuitionistic (α,β) - Fuzzy H_v -submodules

Definition 4.1. [14] Let M_1 and M_2 be two H_v-modules over an H_v-ring R. A mapping f from M_1 into M_2 is called homomorphism if $f(x+y)=f(x)+f(y)$ and $f(r.x)=r.f(x)$, for all $x,y\in M_1$ and $r\in R$.

Definition 4.2. A fuzzy set μ in a set X is said to have sup property if forevery non-empty subset S of X, there exists $x_0\in S$ such that $\mu(x_0)=\sup_{x\in S}\{\mu(x)\}$.

Theorem 4.3. Let M_1 and M_2 be two H_v-modules over an H_v-ring of*R* and mapping ffrom M_1 into M_2 be a surjection. Let $A=\{\tilde{\mu}_A,\tilde{\lambda}_A\}$ is an interval-valuedintuitionistic (α,β)-fuzzy H_v-submodule of M_1 such that $\tilde{\mu}_A$ and $\tilde{\lambda}_A$ have supproperty, then for all $t\in(0,1]$ we have

(1)$U\left(\alpha f\left(\tilde{\mu}_A\right),t\right)=f\left(U\left(\alpha\tilde{\mu}_A,t\right)\right)$,

(2)$L\left(\alpha f\left(\tilde{\lambda}_A\right),t\right)\subseteq f\left(L\left(\alpha\tilde{\lambda}_A,t\right)\right)$,

Where $\alpha\in\{\in,q\}$.

Proof. (1) We only prove the case of $\alpha=\in$. The others are analogous.

$$
\begin{aligned}
& y\in U\left(\in f\left(\tilde{\mu}_A\right),t\right)\\
&\Leftrightarrow y_t\in f\left(\tilde{\mu}_A\right)\\
&\Leftrightarrow f\left(\tilde{\mu}_A\right)(y)\geq t\\
&\Leftrightarrow \sup_{x\in f^{-1}(y)}\left\{\tilde{\mu}_A(x)\right\}\geq t\\
&\Leftrightarrow \exists x'\in f^{-1}(y),\tilde{\mu}_A(x')\geq t\\
&\Leftrightarrow f(x')=y,x'_t\in\tilde{\mu}_A\\
&\Leftrightarrow f(x')=y,x'U\left(\in\tilde{\mu}_A,t\right)\\
&\Leftrightarrow y\in f\left(U\left(\in\tilde{\mu}_A,t\right)\right).
\end{aligned}
$$

(2) We only prove the case of $\beta=q$. The others are analogous.

$$
\begin{aligned}
& y\in L\left(qf\left(\tilde{\lambda}_A\right),t\right)\\
&\Rightarrow y_t\overline{q}f\left(\tilde{\lambda}_A\right)\\
&\Rightarrow f\left(\tilde{\lambda}_A\right)(y)+t\leq 1\\
&\Rightarrow \sup_{x\in f^{-1}(y)}\left\{\tilde{\lambda}_A(x)\right\}+t\leq 1\\
&\Rightarrow \tilde{\lambda}_A(x)+t\leq 1\forall x\in f^{-1}(y)\\
&\Rightarrow x_t\overline{q}\tilde{\lambda}_A\forall x\in f^{-1}(y)\\
&\Rightarrow x\in L\left(q\tilde{\lambda}_A,t\right)\forall x\in f^{-1}(y)\\
&\Rightarrow y\in f\left(L(q\tilde{\lambda}_A,t)\right).
\end{aligned}
$$

Corollary 4.4. Let M_1 and M_2 be two H_v-modules over an H_v-ring of *R* and mapping ffrom M_1 into M_2 be a surjection. Let $A=\left(\tilde{\lambda}^c_A,\tilde{\lambda}_A\right)$ is an interval-valued intuitionistic $(\in,q)$-fuzzy H_v-submodule of M_1 such that $\tilde{\lambda}_A$ have sup property, then for all $t\in(0,1]$ we have $L\left(\alpha f\left(\tilde{\lambda}_A\right),t\right)=U\left(\alpha' f\left(\tilde{\lambda}^c_A\right),t\right)$, where $\alpha\in\{\in,q\}$.

Corollary 4.5. Let M_1 and M_2 be two H_v-modules over an H_v-ring of R and mapping ffrom M_1 into M_2 be a surjection. Let $A=\left(\tilde{\mu}_A,\tilde{\mu}_A^c\right)$ is an interval-valued intuitionistic $(\in,q)$-fuzzy H_v-submodule of M_1 such that $\tilde{\lambda}_A$ have sup property, then for all $t\in(0,1]$ we have $U\left(\alpha f\left(\tilde{\mu}_A\right),t\right)=L\left(\alpha' f\left(\tilde{\mu}_A^c\right),t\right)$, where $\alpha\in\{\in,q\}$.

Theorem 4.6. Let M_1 and M_2 be two H_v-modules over an H_v-ring of R and mapping ffrom M_1 into M_2 be a map. Let $A=\{\tilde{\mu}_A,\tilde{\lambda}_A\}$ is an interval-valued intuitionistic (α,β)-fuzzy H_v-submodule of M_2 such that $\tilde{\mu}_A$ and $\tilde{\lambda}_A$ have sup property, then for all $t\in(0,1]$ we have

(1)$U\left(\alpha f^{-1}\left(\tilde{\mu}_A\right),t\right)=f^{-1}\left(U\left(\alpha\tilde{\mu}_A,t\right)\right)$,

(2)$L\left(\alpha f^{-1}\left(\tilde{\lambda}_A\right),t\right)\subseteq f^{-1}\left(L\left(\alpha\tilde{\lambda}_A,t\right)\right)$,

Where $\alpha\in\{\in,q\}$.

Proof. (1) We only prove the case of $\alpha=\in$. The others are analogous.

$$
\begin{aligned}
& x\in U\left(\in f^{-1}\left(\tilde{\mu}_A\right),t\right)\\
& \Leftrightarrow x_t\in f^{-1}\left(\tilde{\mu}_A\right)\\
& \Leftrightarrow f^{-1}\left(\tilde{\mu}_A\right)(x)\geq t\\
& \Leftrightarrow \tilde{\mu}_A\left(f(x)\right)\geq t\\
& \Leftrightarrow f(x)_t\in\tilde{\mu}_A\\
& \Leftrightarrow f(x)\in U\left(\in\tilde{\mu}_A,t\right)\\
& \Leftrightarrow x\in f^{-1}\left(U\left(\in\tilde{\mu}_A,t\right)\right).
\end{aligned}
$$

(2) We only prove the case of $\beta=q$. The others are analogous.

$$
\begin{aligned}
& x\in L\left(qf^{-1}\left(\tilde{\lambda}_A\right),t\right)\\
& \Rightarrow x_t\overline{q}f^{-1}\left(\tilde{\lambda}_A\right)\\
& \Rightarrow f^{-1}\left(\tilde{\lambda}_A\right)(x)+t\leq 1\\
& \Rightarrow \tilde{\lambda}_A\left(f(x)\right)+t\leq 1\\
& \Rightarrow f(x)_t\,\overline{q}\tilde{\lambda}_A\\
& \Rightarrow f(x)\in L\left(q\tilde{\lambda}_A,t\right)\\
& \Rightarrow x\in f^{-1}\left(L(q\tilde{\lambda}_A,t)\right).
\end{aligned}
$$

Corollary 4.7. Let M_1 and M_2 be two H_v-modules over an H_v-ring of R and mapping ffrom M_1 into M_2 be a map. Let $A=\left(\tilde{\lambda}_A^c,\tilde{\lambda}_A\right)$ is an interval-valued intuitionistic $(\in,q)$-fuzzy H_v-submodule of M_2 such that $\tilde{\lambda}_A$ have sup property, then for all $t\in(0,1]$ we have $L\left(\alpha f^{-1}\left(\tilde{\lambda}_A\right),t\right)=U\left(\alpha' f^{-1}\left(\tilde{\lambda}_A^c\right),t\right)$, where $\alpha\in\{\in,q\}$.

Corollary 4.8. Let M_1 and M_2 be two H_v-modules over an H_v-ring of *R* and mapping ffrom M_1 into M_2 be a map. Let $A=\left(\tilde{\mu}_A,\tilde{\mu}_A^c\right)$ is an interval-valued intuitionistic $\left(\in,q\right)$-fuzzy H_v-submodule of M_2 such that $\tilde{\mu}_A$ have sup property, then for all $t\in(0,1]$ we have $U\left(\alpha f^{-1}\left(\tilde{\mu}_A\right),t\right)=L\left(\alpha' f^{-1}\left(\tilde{\mu}_A^c\right),t\right)$, where $\alpha\in\left\{\in,q\right\}$.

References

[1] B. Davvaz, A brief survey of the theory of H_v-structures, in: Proceedings of the 8th International Congress on AHA, Greece 2002, Spanids Press, 2003, pp. 39-70.
[2] B. Davvaz, W. A. Dudek, Intuitionistic fuzzy H_v–ideals, International Journal of Mathematics and Mathematical Sciences, Volume 2006, Pages 1-11.
[3] B. Davvaz, W. A. Dudek, Y. B. Jun, Intuitionistic fuzzy H_v–submodules, Inform. Sci. 176 (2006) 285-300.
[4] F. Marty, Sur une generalization de la notion de group, in: 8th congress Math.Skandenaves, Stockholm, 1934, pp. 45-49.
[5] K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87-96.
[6] K. T. Atanassov, Intuitionistic fuzzy sets: Theory and Applications, Studies in fuzziness and soft computing, 35, Heidelberg, New York, Physica-Verl., 1999.
[7] K. T. Atanassov, New operations defined over the intuitionistic fuzzy sets, Fuzzy Sets and Systems 61, (1994) 137-142.
[8] L. A. Zadeh, Fuzzy sets, Inform. And Control 8 (1965) 338-353.
[9] M. Asghari-Larimi, Homomorphism of Intuitionistic (α, β)-Fuzzy H_v-Submodules, The Journal of Mathematics and Computer Science 3 (2011), No. 3, 287-300.
[10] P. M. Pu and Y. M. Liu, Fuzzy topology: Neighourhood structure of a fuzzy point and Moore-Smooth convergence, J. Math. Anal. Appl., 76 (1980) 571-599.
[11] S. K. Bhakat, P. Das, $\left(\in,\in\vee q\right)$-fuzzy subgroups, Fuzzy Sets and Systems, 80 (1996) 359-368.
[12] S. K. Bhakat, P. Das, On the definition of a fuzzy subgroup, Fuzzy Sets and Systems, 51 (1992) 235-241.
[13] T. Vougiouklis, A new class of hyperstructures, J. Combin. Inf. System Sci., to appear.
[14] T. Vougiouklis, Hyperstructures and their representations, Hadronic Press, Florida, 1994.
[15] X. H. Yuan , H. X. Li and E. S. Lee, On the definition of the intuitionistic fuzzy subgroups, Computers and Mathematics with Applications, 59 (2010) 3117-3129.
[16] J. Zhan, B. Davvaz, P. Corsini, On Intuitionistic (S,T)-fuzzy H_v-submodules of Hv-Modules, Southeast Asian Bulletin of Mathematics 36 (2012) 589-601.
[17] B. Davvaz, W. A. Dudek, Intuitionistic fuzzy H_v–ideals, International Journal of Mathematics and Mathematical Sciences, 2006, 1-11.
[18] A.K.Sinha, M.K.Dewangan,On Interval-valued Intuitionistic (α, β)-Fuzzy H_v-Submodules,International Journal of Engineering Sciences & Research 4 (2015) No. 7, 333-341.