Basic Concepts on Posbist Reliability Theory

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Abstract- The conventional reliability theory is built on the probability assumption and the binary-state assumption . It has been successfully used for solving various reliability problems. However, it is not suitable when the failure probabilities concerned are very small or when there is a lack of sufficient data. As a result, researchers have been searching for new models and new reliability theories that overcome the shortcomings of the classical probabilistic definition of reliability. Focusing also the possibility Distribution Function of Posbist systems.

Keywords – Posbist systems,profust, conventional reliability

I. INTRODUCTION

Structural reliability analysis differs in many important ways from reliability analysis as practised in, for example, the electronics and aero-space industries, in spite of the fact that the underlying probabilistic nature of the problems are the same. A clear understanding of these differences is helpful and they will be discussed later. Both branches have developed from classical reliability theory - a subject which evolved as a result of the increasing need for reliable electronic systems during the 1940's, initially for military applications.

Structural reliability theory and Its Applications Structural Reliability theory and its Applications.

It is well known that the conventional reliability theory is based on two fundamental assumptions [Barlow and Proschan 1981]:

(A) Probability assumption: the system (failure) behavior can be fully characterized in the context of probability measure.

(B) Binary state assumption: the meaning of system failure is defined precisely and thus at any time the system is in one of the two crisp states—fully functioning state and fully failed state.

But there are cases where either or both of these two assumptions need modifications. The alternative assumptions are :

(A') Possibility assumption: The system failure behavior can be fully characterized in the context of possibility measure.

(B') Fuzzy- state assumption: At any time the system may be, to some extent, in one of two fuzzy states- fuzzy success state and fuzzy failure state. This means, the system failure is not defined precisely, but in a fuzzy way.

It is also called PROBIST reliability theory, since it is based on the probability and binary state assumption. Systems studied in the context of probist reliability theory are called probist systems. Although these two assumptions have been accepted in the past decades(since 1950s) and sound reasonable in extensive cases, it has been strongly argued that they are no longer the case in widerange of cases [Zadeh [1965]. As a consequence, three forms of fuzzy reliability theory are developed and discussed in the literature [Cai 1965]:

(1) **PROFUST:** reliability theory – based on PROBABILITY assumption and FUZZY STATE assumption.

(2)**POSBIST:** reliability theory – based on POSSIBILITY assumption and BINARY STATE assumption.

(3)**PROFUST:** reliability theory – based on POSSIBILITY assumption and FUZZY STATE assumption.

(4)**CONVENTIONAL RELIABILITY :**theory is considered as PROBIST reliability theory based on PROBABILITY assumption and BINARY STATE assumption.

To clarify the concept of fuzziness, [Nahmias1978] proposed a theoretical framework based on fuzzy variable analogous to the random variable in the sample space model of the probability theory. Considering the lifetimes as fuzzy variables, [Kai-Yuan 1991] derived the possibility distribution functions of a two-component series system and a two-component parallel system under the binary state

assumption. Cross considered the problem of evaluation of client-server networks as nested *k*-out-of-*n* systems ($1 \le k \le n$) with fuzzy reliability profiles.[Huang 2004] provided a new technique, called fault tree analysis to characterize posbist system's failure.

II. Basic Concepts of the Posbist Reliability Theory

The concept of the posbist reliability theory is introduced in [Kai-Yuan et al 1981]. Since the binary state assumption is reserved, the failure of a system (or component) is defined precisely and the system lifetime, denoted by *X*, is calculated from the instant the system starts functioning to the instant the system fails. However, since the possibility assumption is taken, the instant of system failure occurrence is characterized in the context of possibility measure. The
lifetimes of a posbist system and its components are taken as nonnegative real-valued fuzzy variables.

The definitions are borrowed from [Kai-Yuan 1981] and [Nahmias 1981] explores a possible axiomatic framework analogous to the sample space model of the probability theory, from which a rigorous theory of fuzziness can be constructed as given below.

Let Γ be an abstract space of generic elements $\gamma \in \Gamma$. The actual construction of Γ will depend upon the particular problem being modeled in much the same way as the construction of the sample space in probability depends upon the particular random experiment.

Definition 1. Suppose that G is the discrete topology on Γ (that is, the class of all subsets of Γ). A scale *σ*, defined on G, satisfies the following axioms:

$$(i)\sigma(\phi)=0, \sigma(\Gamma)=1;$$

$(ii)\sigma(\cup A_\alpha)=\sup\sigma(A_\alpha)$ for any arbitrary collection of sets A_α in g.

The triplet (Γ, g, σ) is called the pattern space.

Definition 2. A fuzzy variable *X* is a real-valued function defined on a pattern space (Γ,g, *σ*).

Definition 3. The possibility distribution function of a fuzzy variable *X*, denoted by $\mu_X(x)$, is a mapping from R to the unit interval [0, 1] and is given by

$$\mu_X(x)=\sigma(\gamma : X(\gamma)=x) \qquad \forall x \in R.$$

Further, *X* is said to be normal (respectively, convex or strictly convex) if the fuzzy set $\{(x,\mu_X(x)) : x \in R^+\}$ is normal (respectively, convex or strictly convex).

Definition 4. Given a pattern space (Γ,g, *σ*), the sets $A_1, A_2, ..., A_n \subset g$ are said to be mutually un-related if $\sigma(A_{i_1} \cap A_{i_2} \cap ....A_{i_k}) = \min(\sigma(A_{i_1}), \sigma(A_{i_2}), ..., \sigma(A_{i_k})), for 1 \le i_1 \ne i_2 \ne ... \ne i_k \le n$ and $1 \le k \le n$.

Definition 5. Given a pattern space (Γ,g, *σ*), the fuzzy variables $X_1, ..., X_n$ are said to be mutually unrelated if the sets $\{X_{i_1} = x_1\}, ..., \{X_{i_k} = x_k\}$ are unrelated for $x_1, ..., x_k \in R$ whenever $1 \le i_1 \ne i_2 \ne ... \ne i_k \le n$ and $1 \le k \le n$.

The lifetime of a posbist system (posbist component) is a nonnegative real-valued fuzzy variable.

Definition 6. Posbist reliability $R(t)$ of a system is the possibility that the system performs the functions properly during a predefined exposure period under a given environment; that is, $R(t)=\sigma(X>t)=\sup_{u>t}\mu_X(u), t\in R^+$

Next, they state a lemma which will be used in proving the upcoming theorems in the next section.

Lemma7. For any sets $A_1, A_2, A_3, ..., A_n \text{ with } A_i \subseteq A_{i+1} \subseteq R$, i = 1, 2, . . . , $\overline{n-1}$, $\max\left(\min_{1\le i\le n} A_i\right)=\min A_{1,}$.

where minimum of a set is defined as the elementwise minimum; that is, for $A=\{a_1,...,a_n\}, \min A=\min_{1\le i\le n} a_{i.}$

They now state a lemma from [Kai-Yuan 1981] to be used in proving the subsequent theorems.

Lemma 8. Suppose X is the system lifetime defined on a possibility space (Γ,g, σ). Suppose that X is a strictly convex

fuzzy variable with the possibility distribution function $\mu_X(x)$ which is continuous. Then there exists a unique point, say x_0 (finite or infinite), such that

$$\sup_{u\ge x}\mu_X(u)=\begin{cases}\mu_X(x_0), x\le x_0\\ \mu_X(x), x>x_0\end{cases}$$

III. THE POSSIBILITY DISTRIBUTION FUNCTION OF POSBIST SYSTEMS

The following theorem gives the possibility distribution function of a 2-out-of-3 system in terms of those of the components, which generalize Kai-Yuan et al.[1981] work giving the possibility distribution a parallel system with two components.

Theorem 9

For a 2-out-of-3 system let X_j denote the lifetime of the jth component , j=1,2,3.They assume that X_j is a normed unrelated and strictly convex fuzzy variable with corresponding continuous possibility distribution function $\mu_{X_j}(x), j=1,2,3$. Let X denote the system lifetime .

Then there exists a set of unique real numbers $\{a_1,a_2,a_3\}, a_i\in R^+$ for i=1,2,3 such that the possibility distribution function of X , denoted by $\mu_X(x)$ is given by

$$\mu_X(x)=\begin{cases}\min\{\mu_{X_1}(x),\mu_{X_2}(x),\mu_{X_3}(x)\} & if\ x\le a_1\le a_2\le a_3\\ \min\{\mu_{X_2}(x)\mu_{X_3}(x)\} & if\ a_1<x\le a_2\le a_3\\ \max\{\min(\mu_{X_2}(x)\mu_{X_3}(x)),\min(\mu_{X_1}(x)\mu_{X_3}(x))\} & if\ a_1\le a_2<x\le a_3\\ \max\{\min(\mu_{X_2}(x)\mu_{X_3}(x)),\min(\mu_{X_1}(x)\mu_{X_3}(x)), & \\ \min(\mu_{X_1}(x)\mu_{X_2}(x))\} & if\ a_1\le a_2\le a_3<x\end{cases}$$

Proof.

$$\mu_X(x) = \sigma(X = x)$$
$$= \sigma(\text{at least two of } X_1, X_2, X_3 = x)$$
$$= \sigma\left[\{X_1 = x, X_2 = x, X_3 = x\} \cup \{X_1 = x, X_2 = x, X_3 \le x\}\right.$$
$$\left.\cup \{X_1 = x, X_3 = x, X_2 \le x\} \cup \{X_3 = x, X_2 = x, X_1 \le x\}\right]$$
$$= \max\left[\sigma\{X_1 = x, X_2 = x, X_3 = x\}, \sigma\{X_1 = x, X_2 = x, X_3 \le x\},\right.$$
$$\left.\sigma\{X_1 = x, X_3 = x, X_2 \le x\}, \sigma\{X_3 = x, X_2 = x, X_1 \le x\}\right]$$

Now for
$1 \le i \ne j \ne k \le 3,$

$$\sigma\left[\{X_i = x, X_j = x, X_k \le x\}\right] = \min\left\{\sigma(X_i = x), \sigma(X_j = x), \sigma(X_k \le x)\right\}$$
$$= \min\left[\mu_{X_i}(x), \mu_{X_j}(x), \sup_{u \le x} \mu_{X_k}(u)\right]$$

and

$$\sigma\left[\{X_1 = x, X_2 = x, X_3 \le x\}\right] = \min\left\{\sigma(X_1 = x), \sigma(X_2 = x), \sigma(X_3 \le x)\right\}$$
$$= \min\left[\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x)\right]$$

without loss of generality, they assume that $a_1 \le a_2 \le a_3$. It is to be noted that, for i=1,2,3,

$$\sup_{u \le x} \mu_{X_i}(u) = \begin{cases} \mu_{X_i}(x) & \text{if } x \le a_i \\ 1 & \text{if } x > a_i \end{cases}$$

Then, for $x \le a_1$, equation (6) gives

$$\mu_X(x) = \min\left(\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x)\right).$$

If $a_1 < x \le a_2$, *we have, for* $i \ne j \ne k$ *and* $k = 2, 3,$

$$\sigma\left[\{X_i = x, X_j = x, X_k \le x\}\right] = \min\left\{\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x)\right\}$$

whereas

$$\sigma\left[\{X_2 = x, X_3 = x, X_1 \le x\}\right] = \min\left\{\mu_{X_2}(x), \mu_{X_3}(x)\right\}.$$

Combining all these observations, equation (6) gives

$$\mu_X(x) = \max\left\{\min\left(\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x)\right), \min\left(\mu_{X_2}(x), \mu_{X_3}(x)\right)\right\}$$
$$= \min\left(\mu_{X_2}(x), \mu_{X_3}(x)\right).$$

$1 \le i \ne j \ne k \le 3,$

$$\sigma\left[\left\{X_i = x, X_j = x, X_k \le x\right\}\right] = \min\left\{\sigma\left(X_i = x\right), \sigma\left(X_j = x\right), \sigma\left(X_k \le x\right)\right\}$$
$$= \min\left[\mu_{X_i}(x), \mu_{X_j}(x), \sup_{u \le x} \mu_{X_k}(u)\right]$$

and

$$\sigma\left[\left\{X_1 = x, X_2 = x, X_3 \le x\right\}\right] = \min\left\{\sigma\left(X_1 = x\right), \sigma\left(X_2 = x\right), \sigma\left(X_3 \le x\right)\right\}$$
$$= \min\left[\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x)\right]$$

without loss of generality, they assume that $a_1 \le a_2 \le a_3$. It is to be noted that, for $i = 1, 2, 3,$

$$\sup_{u \le x} \mu_{X_i}(u) = \begin{cases} \mu_{X_i}(x) & if\ x \le a_i \\ 1 & if\ x > a_i \end{cases}$$

Then, for $x \le a_{1,}$ equation (6) gives

$$\mu_X(x) = \min\left(\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x)\right).$$

If $a_1 < x \le a_2$, we have, for $i \ne j \ne k$ *and* $k = 2, 3,$

$$\sigma\left[\left\{X_i = x, X_j = x, X_k \le x\right\}\right] = \min\left\{\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x)\right\}$$

whereas

$$\sigma\left[\left\{X_2 = x, X_3 = x, X_1 \le x\right\}\right] = \min\left\{\mu_{X_2}(x), \mu_{X_3}(x)\right\}.$$

Combining all these observations, equation (6) gives

$$\mu_X(x) = \max\left\{\min\left(\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x)\right), \min\left(\mu_{X_2}(x), \mu_{X_3}(x)\right)\right\}$$
$$= \min\left(\mu_{X_2}(x), \mu_{X_3}(x)\right).$$

hence proved.

Interestingly, the possibility distribution function as given in Theorem 9 for a *k*-out-of-*n* system is true only when *k* < *n*. For *k* = *n*, the theorem is stated below, which generalize Kai-Yuan et al. [1981] work giving the possibility distribution a series system with two components. The proof is omitted for brevity.

Theorem-10

For a series system, let X_j denote the lifetime of the jth component. Assume that X_j's are normal unrelated and strictly convex fuzzy variables with the corresponding continuous possibility distribution function $\mu_{X_j}(x)$, j = 1, 2, . . . , n.

Let X denote the system lifetime. Then there exists a set of unique real numbers $\{a_1, a_2, ..., a_n\}, a_i \in R^+$ for i=1,2,…,n, such that the possibility distribution function of X, denoted by $\mu_X(x)$,*with* $a_1 \le a_2 \le ... \le a_i < x \le a_{i+1} \le a_{i+2} \le ... \le a_n$ *is given by*

$$\mu_X(x) = \begin{cases} \min\left\{\mu_{X_j}(x) : j = i, i-1, ..., 1\right\} \\ \qquad if\ 1 \le i \le n, \\ \max\left\{\mu_{X_j}(x) : j = 1, 2, ..., n\right\} \\ \quad if\ x < a_1 \le a_2 \le ... \le a_n \end{cases}$$

IV.CONCLUSION

The work done on various posbist systems can be concluded in the following points

(a) The possibility distribution functions of 1– out –of -2 and 2-out-of-2 systems

(cf. Kai-Yuan) can be obtained from the possibility distribution functions of k-out-of-n and n-out-of-n systems by using k=1 and n=2.

(b) In the posbist reliability theory, the possibility distribution function and hence the reliability function of series system, that is , n-out-of–n system cannot be obtained from the possibility distribution function of k-out-of –n system by using $k=n$ whereas in the conventional reliability theory, the reliability function of an n-out-of-n system can be obtained as a special case from the reliability function of a k-out-of-n system.

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