POISSON – GEOMETRIC PROBABILITY DISTRIBUTION USING LAGRANGE EXPANSION

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Abstract

In this paper, we introduced discrete probability distribution named as Poisson - Geometric Distribution (PGD) from Lagrange expansion of second kind for count data applications. The proposed distribution is in the form of Modified Power Series Distribution (MSPD) and some of its characteristics are utilized and Also we expressed the proposed model is in the form of Inflated Modified Power Series Distribution (IMPSD). Maximum Likelihood Estimation (MLE) method is adopted to estimate the parameters of the distribution.

Keywords: Second kind Lagrange expansion, Modified Power Series Distribution, inflated modified power series distribution, maximum likelihood estimation method.

1. Introduction

In this paper, we introduced Poisson – Geometric distribution using Lagrangian expansion of second kind for count data models. Consul and Shenton (1972, 1973) proposed Lagrange expansion for a new class of generating discrete generalized probability distributions. Janardan and Rao (1983) and Janardan (1987, 1997) discussed the class of Lagrange probability distributions based on second kind and also derived weighted distribution function and Lagrange distributions of the second kind. Gupta (1974, 1975) proposed the recurrence relation between the moments, central moments, the factorial moments, etc. of MPSD and Maximum likelihood estimation of a modified power series distribution and some of its applications.Consul and Famoye (2001) proposed a Lagrangian expansion of second kind and its properties. Consul and Famoye (2005) introduced Dev probability distribution using Lagrangian expansion of second kind and some of its applications in queuing theory and stochastic model of epidemics are presented and also a numerical example on absenteeism data is obtained. Consul and Famoye (2006a) proposed a new class of discrete probability distribution and named as Harish probability distribution and he also derived MPSD of Harish probability distribution along with its applications in the branching process and queuing theory. Consul and Famoye (2006b) provided an excellent treatise on Lagrangian Probability distributions in various probabilities generating function. Gupta et al., (1995) obtained the zero inflated modified power series distributions (IMPSD) and its structural properties of the distribution of the sum of independent IMPSD variables maximum likelihood estimation of the parameters of the model is examined and derived the asymptotic variance – covariance matrix of the estimators. Murat and Szynal (1998) proposed the class of inflated modified power series distribution (IMPSD) this class include among other the generalized poisson, the generalized negative binomial, the generalized logarithmic series and the lost games distributions. Kumar (1981) obtained some application of Lagrangian distribution in queuing theory and epidemiology. Li et.al. (2006, 2008) proposed some extension of the Lagrangian probability distributions and obtained mixture distributions based on Lagrangian probability models.

The paper is organized as follows, section 2 defines the proposed modal of the Lagrangian probability distribution and some of its properties including behaviour of Poisson – Geometric distribution with different parameter values. Here probability generating function of Geometric function is x success before the first failure $x = 0,1,2,...$. Section 3 MPSD of Poisson – Geometric distribution and maximum likelihood estimation method is used to estimate the parameters of the distribution. Sections 4 propose inflated modified power series distribution of Poisson – Geometric distribution and some of its properties are obtained. Sections 5 evaluate maximum likelihood estimation method for estimating the parameters. In Section 6, we presents the application of Poisson – Geometric using real data set.

2 POISSON – GEOMETRIC DISTRIBUTION

2.1 Theorem Let $X \sim PGD(\theta, m)$ be a random variable from Poisson - Geometric distribution then the probability mass function (pmf) is given by

$$P(X = x) = (1-\theta)^2 e^{-\theta m} (\theta e^{-\theta})^x \sum_{r=0}^{x} \frac{(x+m)^r}{r!}; \; x = 0,1,2,...; \; 0 < \theta < 1; \; m > 0 \qquad (1)$$

Proof

The probability generating function of Poisson distribution and geometric distribution are given as $e^{\theta(z-1)}$ and $(1-\theta)(1-\theta z)^{-1}$ respectively.

Consider the probability mass function of L_2D is,

$$P(X=x)=\frac{1-g'(1)}{x!}\left[D^x[g^x(z)f(z)]\right]_{z=0};\ x=0,1,2,... \tag{2}$$

Further, we know that $f(z)=f_1(z)f_2(z)$, where $f_1(z)$ is a pgf of Poisson distribution and $f_2(z)$ is a pgf of geometric distribution. According to Consul and Famoye (2005), the function $f(z)$ is defined as

$$f(z)=f_1(z)f_2(z)=e^{\theta m(z-1)}(1-\theta)(1-\theta z)^{-1}$$

And take $g(z)$ as the pgf of Poisson distribution.(i.e.,) $g(z)=e^{\theta(z-1)}$.

Then the Lagrangian expansion of second kind given in (2) becomes

$$1=\sum_{y=0}^{\infty}\frac{u^x(1-g'(1))}{x!}\left[D^x[(g(z)^x f(z)]\right]_{z=0}$$

By using the above expressions, we write

$$1=\sum_{x=0}^{\infty}\frac{(1-\theta)}{x!}D^x\left\{e^{\theta x(z-1)}e^{\theta m(z-1)}(1-\theta)(1-\theta z)^{-1}\right\}\Big|_{z=0}$$

$$1=\sum_{x=0}^{\infty}\frac{(1-\theta)}{x!}D^x\left\{e^{\theta x(z-1)}e^{\theta m(z-1)}(1-\theta)(1-\theta z)^{-1}\right\}\Big|_{z=0}$$

$$1=\sum_{x=0}^{\infty}\frac{(1-\theta)^2}{x!}D^x\left\{e^{(\theta x+\theta m)(z-1)}(1-\theta z)^{-1}\right\}\Big|_{z=0} \tag{3}$$

Using Leibnitz theorem

$$D^x(uv)=\sum_{r=o}^{x}\binom{x}{r}D^{x-r}u\ D^r v$$

Consider,

$$u=(1-\theta z)^{-1}$$

$$Du=(-1)(1-\theta z)^{-1-1}(-\theta)$$

$$D^2 u = (-1)(-1-1)(1-\theta z)^{-1-2}(-\theta)^2$$

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$$D^{x-r} u = (-1)(-1-1)(-1-2)...(-1-x+r+1)(1-\theta z)^{-1-x+r}(-\theta)^{x-r} \qquad (4)$$

Consider,

$$v = e^{(\theta x+\theta m)(z-1)}$$

$$Dv = e^{(\theta x+\theta m)(z-1)}(\theta x+\theta m)$$

$$D^2 v = e^{(\theta x+\theta m)(z-1)}(\theta x+\theta m)^2$$

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$$D^r v = e^{(\theta x+\theta m)(z-1)}(\theta x+\theta m)^r \qquad (5)$$

Substitute equations (4) and (5) in (3), we get

$$1 = \sum_{x=0}^{\infty} \frac{(1-\theta)^2}{x!} \left\{ \begin{array}{l} \sum_{r=0}^{x} \frac{x!}{r!(x-r)!} \\ \quad \times \left((-1)(-1-1)(-1-2)...(-1-x+r+1)(1-\theta z)^{-1-x+r}(-\theta)^{x-r}\right) \\ \quad \times \left(e^{(\theta x+\theta m)(z-1)}(\theta x+\theta m)^r\right) \end{array} \right\} \Bigg|_{z=0}$$

$$1 = \sum_{x=0}^{\infty} \frac{(1-\theta)^2}{x!} \left\{ \sum_{r=0}^{x} \frac{x!}{r!} \left(\frac{(x-r)!}{(x-r)!} (\theta)^x \right) \left(e^{-(\theta x+\theta m)} (x+m)^r \right) \right\}$$

$$1 = \sum_{x=0}^{\infty} \frac{(1-\theta)^2}{x!} \left\{ \sum_{r=0}^{x} \frac{x!(x+m)^r}{r!} \left((\theta e^{-\theta})^x \right) e^{-\theta m} \right\} \qquad (6)$$

Since every term in the summation on the right hand side is positive and the sum of all the terms is unity, it satisfies the definition of probability mass function and it is given by,

$$P(X = x) = (1-\theta)^2 e^{-\theta m} (\theta e^{-\theta})^x \sum_{r=0}^{x} \frac{(x+m)^r}{r!} \quad x = 0,1,2,3,...; \ \theta > 0; \ m > 0 \qquad (7)$$

The above probability mass function is called as Poisson - Geometric distribution (PGD).

2.2 Behaviour of Poisson – Geometric Distribution

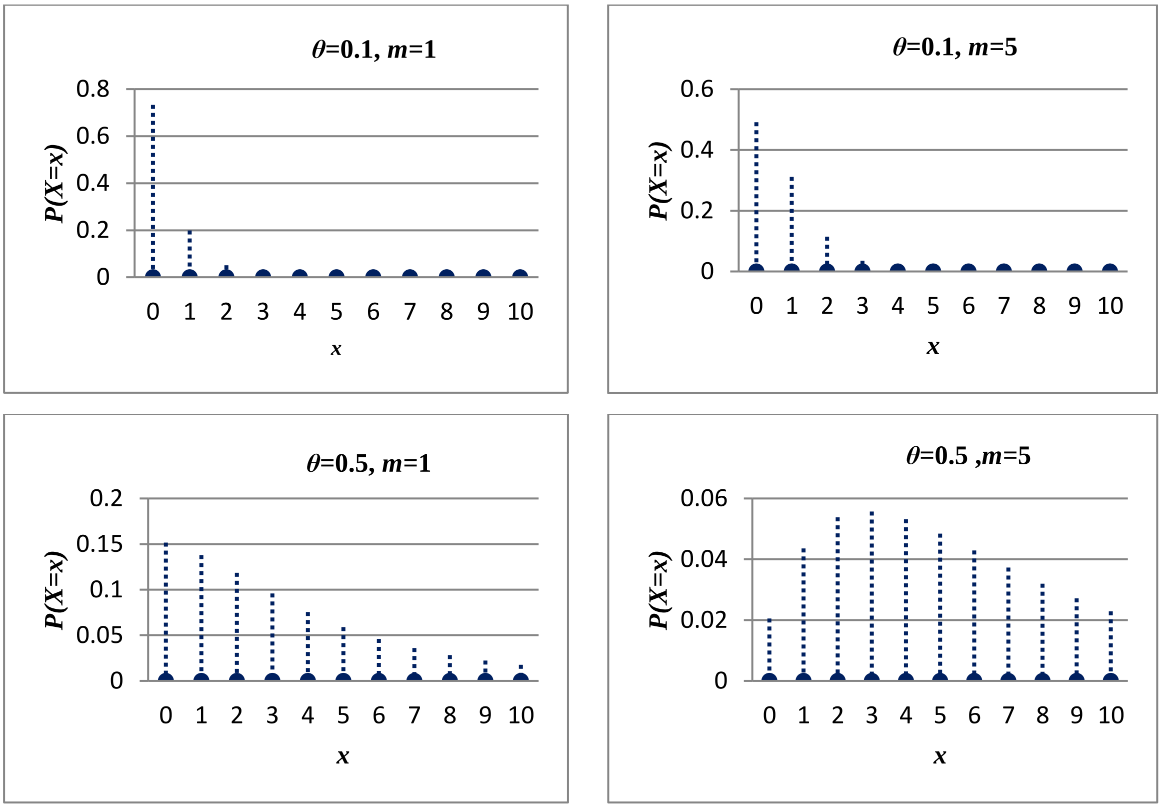

Figure1: Plot of pmf of Poisson – Geometric Distributions for different parameter values θ and m .

3 Modified power series distribution

The form of modified power series distribution (MPSD) is given by Gupta (1974)

$$P(X = x) = a_x \frac{[\phi(\theta)]^x}{h(\theta)}, x \in T \subset N, \tag{9}$$

Where N is the set of non-negative integers and T is a subset of N. All MPSD are linear exponential and form a sub-class of the Lagrangian distribution.

Let take, $g(\theta) = e^{\theta} = g'(\theta)$ and $f(\theta) = (1-\theta)^{-1} e^{m\theta}$ the inversion Lagrangian transformation $\theta = u\, g(\theta) = u\, e^{-\theta}$

Using equation (1), we get

$$\frac{(1-g'(1))\,f(\theta)}{1-\theta g'(\theta)/g(\theta)} = \sum_{x=0}^{\infty} \frac{u^x (1-g'(1))}{x!} \left[D^x[(g(\theta)^x f(\theta)]\right]_{\theta=0}$$

$$\Rightarrow \frac{f(\theta)}{1-\theta g'(\theta)/g(\theta)} = \sum_{x=0}^{\infty} \frac{u^x}{x!} \left[D^x[(g(\theta)^x f(\theta)]\right]_{\theta=0}$$

$$\Rightarrow \frac{(1-\theta)^{-1}e^{\theta m}}{1-\theta e^{\theta}/e^{\theta}} = \sum_{x=0}^{\infty}\frac{\left(\theta e^{-\theta}\right)^{x}}{x!}D^{x}[e^{\theta x}e^{\theta m}(1-\theta)^{-1}]\Big|_{\theta=0}$$

$$= \sum_{x=0}^{\infty}\frac{\left(\theta e^{-\theta}\right)^{x}}{x!}D^{x}[e^{\theta x+m\theta}(1-\theta)^{-1}]\Big|_{\theta=0} \qquad (10)$$

Using Leibnitz theorem

$$(uv)^{x} = \sum_{r=0}^{x}\binom{x}{r}D^{x-r}(u)D^{r}(v)$$

Consider,

$$u = (1-p)^{-1}$$

$$Du = (-1)(1-p)^{-1-1}(-1)$$

$$D^{2}u = (-1)(-1-1)(1-p)^{-1-2}(-1)^{2}$$

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$$D^{x-r}u = (-1)(-1-1)(-1-2)...(-1-x+r+1)(1-p)^{-k+1-x+r}(-1)^{x-r} \qquad (11)$$

Consider,

$$v = e^{x\theta+m\theta}$$

$$Dv = e^{x\theta+m\theta}(x\theta+m\theta)$$

$$D^{2}v = e^{x\theta+m\theta}(x\theta+m\theta)^{2}$$

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$$D^{r}v = e^{x\theta+m\theta}(x\theta+m\theta)^{r} \qquad (12)$$

Substitute equation (11) and (12) in equation (10), we get

$$\frac{(1-\theta)^{-1}e^{\theta m}}{1-\theta} = \sum_{x=0}^{\infty}\frac{\left(\theta e^{-\theta}\right)^{x}}{x!}\left\{\sum_{r=0}^{x}\frac{x!}{r!(x-r)!}\Big((-1)(-1-1)(-1-2)...(-1-x+r+1)(1-p)^{-k+1-x+r}(-1)^{x-r}\Big) \times \Big(e^{x\theta+m\theta}(x\theta+m\theta)^{r}\Big)\right\}\Bigg|_{z=0}$$

Using binomial expansion we get,

$$\frac{(1-\theta)^{-1}e^{\theta m}}{1-\theta} = \sum_{x=0}^{\infty}\left(\theta e^{-\theta}\right)^{x}\sum_{r=0}^{x}\frac{x+m}{r!} \qquad (13)$$

According to definition of MPSD we write,

$$a_x = \sum_{r=0}^{x} \frac{x+m}{r!}$$

$$h(\theta) = (1-\theta)^{-2} e^{m\theta}$$

$$h'(\theta) = -2(1-\theta)^{-2}(-1)e^{m\theta} + (1-\theta)^{-2} e^{m\theta} m$$

$$\phi(\theta) = \theta e^{-\theta}$$

$$\phi'(\theta) = e^{-\theta} + \theta e^{-\theta}(-1)$$

$$\phi'(\theta) = e^{-\theta}(1-\theta)$$

Mean

$$E(X) = \frac{\phi(p)}{\phi'(p)} \frac{h'(p)}{h(p)}$$

$$= \frac{\theta e^{-\theta}}{e^{-\theta}(1-\theta)} \left[\frac{-2(1-\theta)^{-2-1}(-1)e^{\theta m} + (1-\theta)^{-2} e^{m\theta} m}{(1-\theta)^{-2} e^{m\theta}} \right]$$

$$= \frac{\theta}{(1-\theta)} \left[\frac{-2(1-\theta)^{-2-1}(-1)e^{\theta m}}{(1-\theta)^{-2} e^{m\theta}} + \frac{(1-\theta)^{-2} e^{m\theta} m}{(1-\theta)^{-2} e^{m\theta}} \right]$$

$$= \frac{\theta}{(1-\theta)} \left[\frac{2}{1-\theta} + m \right]$$

$$E(X) = \left[\frac{2\theta}{(1-\theta)^2} + \frac{m}{1-\theta} \right] \tag{14}$$

Variance

$$\sigma^2 = \mu_2 = \frac{\phi(\theta)}{\phi'(\theta)} \frac{d\mu}{d\theta}$$

$$\frac{d\mu}{d(\theta)} = \left[\left(\frac{m}{1-\theta} \right) + \left(\frac{2\theta}{(1-\theta)^2} \right) \right]$$

$$= \left[m\left[\frac{(1-\theta)-\theta(-1)}{(1-\theta)^2} \right] + 2\left[\frac{(1-\theta)^2 + \theta(2(1-\theta))}{(1-\theta)^4} \right] \right]$$

$$= \left[\frac{m}{(1-\theta)^2} + 2\left[\frac{(1-\theta)((1-\theta)+2\theta)}{(1-\theta)^4} \right] \right]$$

$$= \frac{m}{(1-\theta)^2} + 2\left[\frac{(1-\theta)+2\theta}{(1-\theta)^3} \right]$$

$$= \frac{m}{(1-\theta)^2} + \frac{2(1-\theta)}{(1-\theta)^3}$$

$$\sigma^2 = \mu_2 = \frac{\phi(\theta)}{\phi'(\theta)} \frac{d\mu}{d\theta}$$

$$= \frac{\theta e^{-\theta}}{(1-\theta)e^{-\theta}} \left[\frac{m}{(1-\theta)^2} + \frac{2(1-\theta)}{(1-\theta)^3} \right]$$

$$= \frac{\theta}{(1-\theta)} \left[\frac{m}{(1-\theta)^2} + \frac{2(1-\theta)}{(1-\theta)^3} \right]$$

$$\sigma^2 = \left[\frac{m\theta}{(1-\theta)^3} + \frac{2(1-\theta)\theta}{(1-\theta)^4} \right] \tag{15}$$

$$\mu_{r+1} = \frac{\phi(p)}{\phi'(p)} \frac{d\mu_r}{d(p)} + r\mu_2\mu_{r-1}, r = 2,3,... \tag{16}$$

Where $\mu_0 = 1,\ \mu_1 = 0,$ and $\phi(p), h'(p)$ denote the derivatives of $\phi(p)$ and $h(p)$ respectively with respect to p. Since $0 < \theta$ and $\sigma^2 - \mu > 0$ the variance σ^2 is larger than the mean μ for all values of the parameters m and θ.

3.1 Recurrence relation of moments

Let the j-th moment about the origin be denoted by μ'_j so that $\mu'_0 = 1$

$$\mu'_j = \sum_{x=0}^{\infty} x^j P(X = x) \tag{17}$$

$$\mu'_j = \sum_{x=0}^{\infty} x^j (1-\theta)^2 e^{-\theta m} (\theta e^{-\theta})^x \sum_{r=0}^{x} \frac{(x+m)^r}{r!}$$

On differentiation both side with respect to θ

$$\frac{d\mu'_j}{d\theta} = \sum_{x=0}^{\infty} x^j \left[2(1-\theta)^{2-1}(-1)\right] e^{-\theta m} (\theta e^{-\theta})^x \sum_{r=0}^{x} \frac{(x+m)^r}{r!}$$

$$+ \sum_{x=0}^{\infty} x^j (1-\theta)^2 \left[e^{-\theta m} - m\right] (\theta e^{-\theta})^x \sum_{r=0}^{x} \frac{(x+m)^r}{r!}$$

$$+ \sum_{x=0}^{\infty} x^j (1-\theta)^2 e^{-\theta m} \left[x(\theta e^{-\theta})^{x-1}\left(e^{-\theta} - \theta e^{-\theta}\right)\right] \sum_{r=0}^{x} \frac{(x+m)^r}{r!}$$

$$\frac{d\mu'_j}{d\theta} = \frac{-2}{1-\theta} \mu_{j'} - m\mu'_j + \frac{e^{-\theta}(1-\theta)}{e^{-\theta}\theta} \mu'_{j+1}$$

$$\frac{(1-\theta)}{\theta} \mu'_{j+1} = \frac{d\mu'_j}{d\theta} + \frac{2}{(1-\theta)} \mu'_j + m\mu'_j$$

$$\mu'_{j+1} = \frac{\theta}{1-\theta}\frac{d\mu'_j}{d\theta} + \frac{2\theta}{(1-\theta)^2}\mu'_j + \frac{m\theta}{(1-\theta)\theta}\mu'_j$$

$$\mu'_{j+1} = \frac{\theta}{1-\theta}\frac{d\mu'_j}{d\theta} + \left[\frac{2\theta}{(1-\theta^2)} + \frac{m\theta}{1-\theta}\right]\mu'_j$$

$$\mu'_{j+1} = \frac{\phi(\theta)}{\phi'(\theta)}\frac{d\mu'_j}{d\theta} + \mu'_1\mu'_j \tag{18}$$

For each values of $j = 0,1,2,...$, one can easily get the values of all required moments .

3.2 Theorem Let X_i, $i = 1,2,3,...,n$ be n independent and identically distributed random variables having the Poisson - Geometric distribution with parameter (m,θ) and its pmf is in the form of the equation given in (3). The distribution of the sample sum $Y = \sum_{i=1}^{n} X_i$ is also Poisson-Geometric distribution with parameters nm and θ.

Proof: Since the pgf of the Poisson-Geometric is given by

$$E[u^X] = H(u) = e^{m\theta(z-1)}(1-\theta)(1-\theta z)^{-1}, \text{Where } z = ue^{\theta(z-1)}$$

Let $Y = \sum_{i=1}^{n} X_i$ be the sum of iid Poisson – Geometric random variables then the pgf of the random variable Y becomes

$$E[u^Y] = E[u^{X_1+X_2+X_3+...+X_n}] = \prod_{i=1}^{n}[u^{X_i}] = \prod_{i=1}^{n} H(u) = [H(u)]^n$$

$$= e^{nm\theta(z-1)}(1-\theta)^n(1-\theta z)^{-n} \tag{19}$$

Where $z = u\,e^{\theta(z-1)}$

3.3 Maximum Likelihood Estimation

The log-likelihood function for the Poisson - geometric probability model can be written in the form

$$\ln L = 2n\ln(1-\theta) - nm\theta + n\bar{x}[\ln(\theta) - \theta] + \sum_{i=1}^{n}\ln\left[\sum_{r=0}^{x_i}\frac{(x_i+m)^r}{r!}\right] \tag{20}$$

On differentiating the above with respect to θ and m. We have the maximum likelihood (ML) equation as,

$$\frac{\partial \log L}{\partial\theta} = 0, \frac{\partial \log L}{\partial m} = 0$$

$$\frac{\partial \log L}{\partial \theta} = -2n(1-\theta)^{-1} - nm + n\bar{x}\left[\theta^{-1} - 1\right] = 0 \tag{21}$$

$$\frac{\partial \log L}{\partial m} = -n\theta + \frac{\partial}{\partial m}\left\{\sum_{i=1}^{n} \ln\left[\sum_{r=0}^{x_i} \frac{(x+m)^r}{r!}\right]\right\} = 0 \tag{22}$$

The value of ML estimates of θ and m can be obtained by solving the above two equations numerically.

4 Inflated modified power series distribution [IMPSD]

A discrete random variable said to have a modified power series distribution inflated at the point π (IMPSD) then it pmf is given by [Gupta (1995)]

$$P(Y=0) = \pi + (1-\pi)\frac{a(0)}{h(\theta)}$$

$$P(Y=y) = \pi + (1-\pi)\frac{a(y)(\phi(\theta))^y}{h(\theta)}; \; y = 1,2,3,... \; 0 < \pi \le 1$$

(23)

4.1 Theorem Let $Y \sim IPGD\ (\theta, m, \pi)$ (i.e.,) Y be inflated Poisson Geometric Distribution and it can be expressed as Inflated Modified Power Series Distribution as given below

$$P(Y=0) = \pi + (1-\pi)\frac{1}{(1-\theta)^{-2} e^{m\theta}}$$

$$P(Y=y) = (1-\pi)\sum_{r=0}^{y} \frac{(y+m)^r}{r!} \frac{\left(\theta e^{-\theta}\right)^y}{(1-\theta)^{-2} e^{m\theta}} \quad ; y = 1,2,3,.. \tag{24}$$

Proof

Consider the inflated modified power series distribution for IPGD [Gupta (1995)].

$$\phi(\theta) = \theta e^{-\theta},\ h(\theta) = (1-\theta)^{-2} e^{m\theta}$$

$$a(x) = \sum_{r=0}^{y} \frac{(y+m)^r}{r!}$$

Hence the proof.

4.2 Behaviour of the Poisson – Geometric distribution when it is in the form Inflated Modified Power Series Distribution

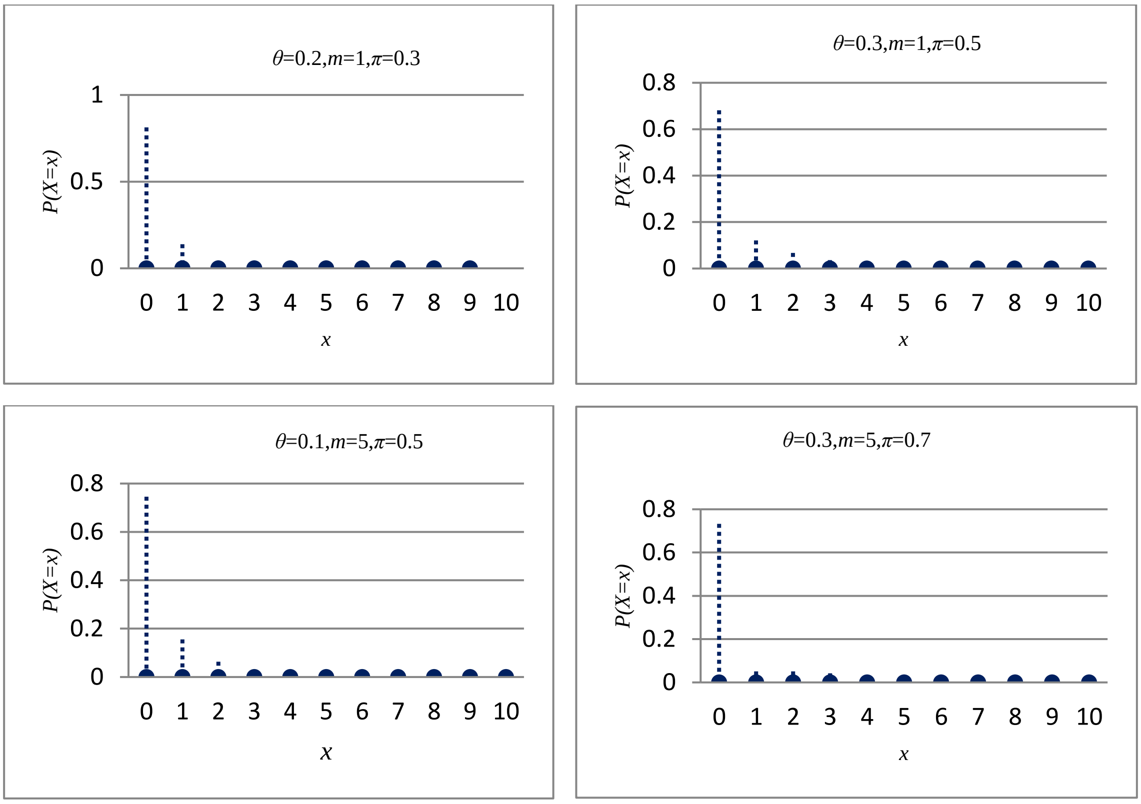

Figure 2. Plot for pmf of Inflated Poisson – Geometric Distribution for different parameter values θ and m and π .

Mean and variance of the IPGD Let $\mu_r{}'$ *and* $\gamma_r{}'$ are the non-central moments of MPSD and IMPSD respectively then

$$\gamma_r' = (1-\pi)\mu_r'; \quad r = 1,2,3,... \tag{25}$$

$$E(Y) = (1-\pi)E(X)$$

$$E(Y) = (1-\pi)\left[\frac{2\theta}{(1-\theta)^2} + \frac{m\theta}{1-\theta}\right] \tag{26}$$

$$V(Y) = (1-\pi)\left[\mathrm{var}(X) + \pi\mu_1'^2\right]$$

$$V(Y) = (1-\pi)\left[\left[\frac{m\theta}{(1-\theta)^3} + \frac{2\theta(1+\theta)}{(1-\theta)^4}\right] + \pi\left[\frac{2\theta}{(1-\theta)^2} + \frac{m\theta}{1-\theta}\right]^2\right] \tag{27}$$

If, these $Var(Y) > Var(X)$ iff $\phi < 1-\alpha^2$ where α is the coefficient of variance of X .

4.3 Probability Generating Function

Let X and Y be a random variable from Poisson – geometric distribution and Inflated Poisson – geometric distribution function respectively. Consider $G_X(t)$ and $G_Y(t)$ be the probability generating function of X and Y respectively.

$$\begin{aligned} G_Y(t) &= \pi + (1-\pi) G_X(t) \\ &= \pi + (1-\pi) e^{(m\theta)(z-1)} (1-\theta)(1-\theta z)^{-1} \end{aligned} \tag{28}$$

The recurrence relation between the noncentral moments of Y can be obtained as follows

$$\begin{aligned} \gamma'_{r+1} &= (1-\pi)\mu'_{r+1} \\ (1-\pi)\gamma'_{r+1} &= (1-\pi)\frac{\phi(\theta)}{\phi^{'}(\theta)}\frac{d}{d\theta}(\gamma'_r(\theta)) + \gamma'_r\,\gamma'_1 \end{aligned} \tag{29}$$

5 Maximum likelihood estimation for IPGD

In this study, the parameters of the IPG distribution are obtained by the maximum likelihood estimation method.

Consider an indicator function as follows.

$$I(y) = \begin{cases} 1;\, y = 0 \\ 0;\, y \in I^+ \end{cases} \tag{30}$$

The likelilhood function of the IPG distribution can be written as

$$L(\pi, m, \theta) = \prod_{i=1}^{n} \{ I(y) P(y=0) + [1 - I(y)] P(y) \neq 0 \} \tag{31}$$

That is,

$$L(\pi,m,\theta) = \prod_{i=1}^{n} \left\{ I \left[\pi + (1-\pi)\frac{1}{(1-\theta)^{-2} e^{m\theta}} \right] + (1-I) \left[(1-\pi) \sum_{r=0}^{y} \frac{(y+m)^r}{r!} \frac{(\theta e^{-\theta})^y}{(1-\theta)^{-2} e^{m\theta}} \right] \right\}$$

By taking,

$$A = \left[\pi + (1-\pi)\frac{1}{(1-\theta)^{-2} e^{m\theta}} \right]$$

$$B = (1-\pi) \sum_{r=0}^{y} \frac{(y+m)^r}{r!} \frac{(\theta e^{-\theta})^y}{(1-\theta)^{-2} e^{m\theta}}$$

Then the log-likelihood function can be written as follows,

$$l(\pi, m, \theta) = \log L(\pi, m, \theta) = \sum_{i=1}^{n} \log\{IA + (1-I)B\}$$

The first order partial derivatives of the likelihood function $l(\pi, m, \theta)$ of the IPG distribution with respect to the parameters π, m and θ of the IPG distribution with respect to the parameters π, m and θ are given below

$$\frac{\partial}{\partial \theta} l(\pi, m, \theta) = \sum_{i=1}^{n} \frac{1}{IA + (1-I)B} \times \frac{\partial}{\partial \theta}\{IA + (1-I)B\} \quad (32)$$

Consider,

$$\frac{\partial}{\partial \theta} IA = \frac{\partial}{\partial \theta} I\left(\pi + (1-\pi)\frac{1}{e^{m\theta}(1-\theta)^{-2}}\right)$$

$$= I(1-\pi)\frac{\partial}{\partial \theta}\left\{e^{-m\theta}(1-\theta)^{2}\right\}$$

$$= I(1-\pi)\left[e^{-m\theta} 2(1-\theta)(-1) + (1-\theta)^{2} e^{-m\theta}(-m)\right]$$

$$= I(1-\pi)e^{-m\theta}\left[-2 + 2\theta - m(1-\theta)^{2}\right]$$

$$= I(1-\pi)e^{-m\theta}\left[2(\theta-1) - m(1-\theta)^{2}\right] \quad (33)$$

Consider,

$$\frac{\partial}{\partial \theta}(1-I)B = \frac{\partial}{\partial \theta}\left[(1-I)(1-\pi)\sum_{r=0}^{y}\frac{(y+m)^{r}}{r!}\frac{(\theta e^{-\theta})^{y}}{(1-\theta)^{-2} e^{m\theta}}\right]$$

$$= (1-I)(1-\pi)\frac{\partial}{\partial \theta}\left[\sum_{r=0}^{y}\frac{(y+m)^{r}}{r!}\frac{(\theta e^{-\theta})^{y} e^{-m\theta}}{(1-\theta)^{-2}}\right]$$

$$= (1-I)(1-\pi)\sum_{r=0}^{y}\frac{(y+m)^{r}}{r!}\left[\frac{(1-\theta)^{-2}\left[(\theta e^{-\theta})^{y} e^{-m\theta}(-m) + e^{-m\theta} y(\theta e^{-\theta})^{y-1}\left[\theta e^{-\theta}(-1) + e^{-\theta}\right]\right] - (\theta e^{-\theta})^{y} e^{-m\theta}(-2)(1-\theta)^{-3}(-1)}{\left[(1-\theta)^{-2}\right]^{2}}\right]$$

$$= (1-I)(1-\pi)\sum_{r=0}^{y}\frac{(y+m)^{r}}{r!}\left[\frac{(1-\theta)^{-2}\left[e^{-m\theta}(\theta e^{-\theta})^{y}(-m) + e^{-m\theta} y(\theta e^{-\theta})^{y-1} e^{-\theta}(1-\theta)\right] - (\theta e^{-\theta})^{y} e^{-m\theta} 2(1-\theta)^{-3}}{\left[(1-\theta)^{-2}\right]^{2}}\right]$$

$$=(1-I)(1-\pi)\sum_{r=0}^{y}\frac{(y+m)^r}{r!}\left[\frac{\begin{array}{l}(1-\theta)^{-2}e^{-m\theta}\left(\theta e^{-\theta}\right)^{y}\left[y(1-\theta)e^{-\theta}\left(\theta e^{-\theta}\right)^{-1}-m\right]\\ -\left(\theta e^{-\theta}\right)^{y}e^{-m\theta}2(1-\theta)^{-3}\end{array}}{(1-\theta)^{-4}}\right]$$

$$=(1-I)(1-\pi)\sum_{r=0}^{y}\frac{(y+m)^r}{r!}e^{-m\theta}\left(\theta e^{-\theta}\right)^{y}(1-\theta)^{-2}\left[\frac{y(1-\theta)e^{-\theta}\left(\theta e^{-\theta}\right)^{-1}-m-2(1-\theta)^{-1}}{(1-\theta)^{-4}}\right]$$

(34)

Substitute equation (31), (32) in equation (30) we write

$$\frac{\partial}{\partial\theta}l(\pi,m,\theta)=\sum_{i=1}^{n}\frac{1}{IA+(1-I)B}\times\left[\begin{array}{l}I(1-\pi)e^{-m\theta}\left[2(\theta-1)-m(1-\theta)^{2}\right]\\ +(1-I)(1-\pi)\\ \sum_{r=0}^{y}\frac{(y+m)^r}{r!}e^{-m\theta}\left(\theta e^{-\theta}\right)^{y}(1-\theta)^{-2}\left[\frac{y(1-\theta)e^{-\theta}\left(\theta e^{-\theta}\right)^{-1}-m-2(1-\theta)^{-1}}{(1-\theta)^{-4}}\right]\end{array}\right]$$

$$\frac{\partial}{\partial m}l(\pi,m,\theta)=\sum_{i=1}^{n}\frac{1}{IA+(1-I)B}\frac{\partial}{\partial m}(IA+(i-I)B) \tag{35}$$

Consider

$$\frac{\partial IA}{\partial m}=\frac{\partial}{\partial m}I\left[\pi+(1-\pi)\frac{1}{e^{m\theta}(1-\theta)^{-2}}\right]$$

$$=(1-\pi)(1-\theta)^{2}\frac{\partial}{\partial m}e^{-m\theta}$$

$$=(1-\pi)(1-\theta)^{2}\theta e^{-m\theta}(-\theta)$$

$$\frac{\partial IA}{\partial m}=-(1-\pi)(1-\theta)^{2}\theta e^{-m\theta} \tag{36}$$

$$\frac{\partial(1-I)B}{\partial m}=\frac{\partial}{\partial m}\left[(1-I)(1-\pi)\sum_{r=0}^{y}\frac{(y+m)^r}{r!}\frac{\left(\theta e^{-\theta}\right)^{y}}{(1-\theta)^{-2}e^{m\theta}}\right]$$

$$=(1-I)(1-\pi)\frac{\left(\theta e^{-\theta}\right)^{y}}{(1-\theta)^{-2}}\frac{\partial}{\partial m}\sum_{r=0}^{y}\frac{(y+m)^r}{r!}\frac{1}{e^{m\theta}}$$

$$=(1-I)(1-\pi)\frac{\left(\theta e^{-\theta}\right)^{y}}{(1-\theta)^{-2}}\left[\sum_{r=0}^{y}\frac{1}{r!}\left[(y+m)^{r}e^{-m\theta}\right]\right]$$

$$=(1-I)(1-\pi)\frac{\left(\theta e^{-\theta}\right)^{y}}{(1-\theta)^{-2}}\left[\sum_{r=0}^{y}\frac{1}{r!}\left[(y+m)^{r}\, e^{-m\theta}(-\theta)+e^{-m\theta} r\,(y+m)^{r-1}\right]\right]$$

$$=(1-I)(1-\pi)\frac{\left(\theta e^{-\theta}\right)^{y}}{(1-\theta)^{-2}}$$

$$\frac{\partial(1-I)B}{\partial m}=(1-I)(1-\pi)\frac{\left(\theta e^{-\theta}\right)^{y}}{(1-\theta)^{-2} e^{m\theta}}\sum_{r=0}^{y}\frac{e^{-m\theta}\,(y+m)^{r}}{r!}\left[r(y+m)^{-1}-\theta\right] \tag{37}$$

Substitute equation (34) and (35) in (33), we write

$$\frac{\partial}{\partial m} l(\pi,m,\theta)=\sum_{r=0}^{n}\frac{1}{IA+(1-I)B}\left[\begin{array}{l} -(1-\pi)(1-\theta)^{2}\theta e^{-m\theta}+(1-I)(1-\pi)\dfrac{\left(\theta e^{-\theta}\right)^{y}}{(1-\theta)^{-2}} \\ \times\displaystyle\sum_{r=0}^{y} e^{-m\theta}\frac{(y+m)^{r}}{r!}\left[r\,(y+m)^{-1}-\theta\right] \end{array}\right]$$

$$\frac{\partial}{\partial \pi} l(\pi,m,\theta)=\sum_{i=1}^{n}\frac{1}{IA+(1-I)B}\frac{\partial}{\partial \pi}\left(IA+(i-I)B\right) \tag{38}$$

Consider,

$$\frac{\partial}{\partial \pi} IA=\frac{\partial}{\partial \pi} I\left[\pi+(1-\pi)\frac{1}{e^{m\theta}(1-\theta)^{-2}}\right]$$

$$=\frac{\partial}{\partial \pi}(I\pi)+\frac{\partial}{\partial \pi} I(1-\pi)\frac{1}{e^{m\theta}(1-\theta)^{-2}}$$

$$\frac{\partial IA}{\partial \pi}=I\left[1-\frac{1}{e^{m\theta}(1-\theta)^{-2}}\right] \tag{39}$$

Consider,

$$\frac{\partial}{\partial \pi}(1-I)B=\frac{\partial}{\partial \pi}(1-I)(1-\pi)\sum_{r=0}^{y}\frac{(y+m)^{r}}{r!}\frac{\left(\theta e^{-\theta}\right)^{y}}{(1-\theta)^{-2} e^{m\theta}}$$

$$\frac{\partial}{\partial \pi}(1-I)B=-(1-I)\sum_{r=0}^{y}\frac{(y+m)^{r}}{r!}\frac{\left(\theta e^{-\theta}\right)^{y}}{(1-\theta)^{-2} e^{m\theta}} \tag{40}$$

Substituting equations (39) and (40) in (38), we write

$$\frac{\partial}{\partial \pi} l(\pi,m,\theta)=\sum_{r=0}^{n}\frac{1}{IA+(1-I)B}\times I\left[1-\frac{1}{e^{m\theta}(1-\theta)^{-2}}\right]$$

The above three partial derivative equations cannot be solved analytically when they equate to zero. Therefore, one adopted Newton – Raphson iterative method to get the estimates of the maximum likelihood estimators.

6 Application

The classic text on probability theory by Feller (1957) includes a number of examples of observations fitting the Poisson distribution, including data on the number of flying-bomb hits in the south of London during World War II. The city was divided into 576 small areas of one-quarter square kilometres each, and the number of areas hit exactly k times was counted. There were a total of 537 hits. The observed frequencies in below Table

Data set 1: Flying – bomb Hits on London During World War II.

Number of bores Per plant	Observed frequency	Expected frequency	
		Poisson distribution	Poisson geometric distribution
0	229	211.33	212.66
1	211	196.29	195.02
2	93	91.16	90.58
3	35	28.22	28.40
4	7	6.55	6.76
5+	1	1.21	1.30
Total	535	535.0	535.0
Estimation of parameter		$\hat{\theta} = 0.928$	$\hat{\theta} = 0.0059$ $\hat{m} = 153.59$
χ^2		4.2503	0.671
d.f		3	1
p-value		0.23	0.24

It is observed from above table that Poisson - geometric distribution (PGD) gives better fit compared to Poisson distribution in terms of chi-squared and p-value.

7 Conclusion

We introduced discrete probability distribution using Lagrange expansion of second kind and it is named as Poisson - geometric distribution. Since the proposed model is in th form of Modified Power Series Distribution (MSPD) and hence some of its characteristics are obtained using MSPD. We showed about the proposed model is also in the form of Inflated Modified Power Series Distribution (IMPSD). Finally we presented an application and showed that PGD provides better fit compare to Poisson distribution for a real data set.

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